

Lie symmetry analysis of the Grad-Shafranov equation

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Abstract

The theory of plasma physics offers a number of nontrivial examples of partial differential equations, which can be successfully treated with symmetry methods. We propose the Grad-Shafranov equation which may illustrate the reciprocal advantage of this interaction between plasma physics and symmetry techniques. A symmetry classification of the Grad-Shafranov equation with two arbitrary functions $F(u)$ and $G(u)$ of the unknown variable $u = u(x, t)$ is given. The optimal system of one-dimensional subalgebras is performed. This latter provides a process for building new solutions for the equation.

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1 Introductction

The equation we are going to examine contains indeed two arbitrary functions $F(u), G(u)$ of unknown variable $u = u(x, t)$, and the goal is now to perform the symmetry classification of this equation. i.e. to find those F, G for which the equation admits nontrivial symmetries. In general, the symmetry propoerties of an equation may strongly depend on the choice of the arbitrary functions involved.

The PDE we want to consider is

$$u_{xx} + a\frac{1}{x}u_x + u_{tt} = x^p F(u) + G(u) \quad (1)$$

with $a = -1, b = 1$ this equation is known in plasma physics as the Grad-Shafranov equation(see [?]) and describes the magnetohydrodynamic force balance in a confined toroidal plasma. In this context, u is the so-called magnetic

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flux variable, x is a radial variable, then $x \geq 0$, while the two arbitrary flux functions $F(u), G(u)$ are related to the plasma pressure and current density profiles.

This equation admits well known properties and a rich literature is devoted to it. After some comments on the general peculiarities of the equation and of its symmetry properties, we shall provide the algebra of its exact Lie point symmetries and probe some properties of its algebra. Next, we shall study one-dimensional optimal system of subalgebras.

According to the standard definitions and procedure (see [?]), we can now look for the equivalence group; preliminary, we look for the full groups of the equation (2), i.e. the intersection of all groups admitted by (2) for any arbitrary choice of F and G . It turns out that this kernel is almost trivial: it contains indeed only the transformation of the variable t .

First of all, we exclude from our classification the case $a = 0$ and $p = 0$ (or $a = 0$ and $F(u) \equiv 0$) because in this case our equation coincides with well known nonlinear Laplace equation.

Now, we investigate the case, $F = 1$ and $G = 0$ for the Grad-Shafranov equation.

2 Symmetry methods

Let a partial differential equation consists p independent and q dependent variables. The one-parameter Lie group of transformations

$$\bar{x}_i = x_i + s\xi^i(x, u) + O(s^2); \quad \bar{u}_\alpha = u_\alpha + s\varphi^\alpha(x, u) + O(s^2), \quad (2)$$

where $\xi^i = \frac{\partial \bar{x}_i}{\partial s}|_{s=0}, i = 1, \dots, p$, and $\varphi^\alpha = \frac{\partial \bar{u}_\alpha}{\partial s}|_{s=0}, \alpha = 1, \dots, q$, are given. The action of the Lie group can be recovered from that of its associated infinitesimal generators. We consider general vector field

$$Y = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^q \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \quad (3)$$

on the space of independent and dependent variables. Therefore, the characteristic of the vector field Y given by (3) is the function

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha(x, u) - \sum_{i=1}^p \xi^i(x, u) \frac{\partial u^\alpha}{\partial x_i}, \quad \alpha = 1, \dots, q. \quad (4)$$

The second prolongation of the infinitesimal generator

$$X = \xi^1(x, t, u) \frac{\partial}{\partial x} + \xi^2(x, t, u) \frac{\partial}{\partial t} + \varphi(x, t, u) \frac{\partial}{\partial u} \quad (5)$$

is the following vector field

$$X^{(2)} = X + \varphi^x \frac{\partial}{\partial u_x} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{xt} \frac{\partial}{\partial u_{xt}} + \varphi^{tt} \frac{\partial}{\partial u_{tt}}$$

with coefficients

$$\begin{aligned}
\varphi^x &= D_x Q + \xi^1 u_{xx} + \xi^2 u_{xt}, \\
\varphi^t &= D_t Q + \xi^1 u_{xt} + \xi^2 u_{tt}, \\
\varphi^{xx} &= D_x^2 Q + \xi^1 u_{xxx} + \xi^2 u_{xxt}, \\
\varphi^{xt} &= D_x D_t Q + \xi^1 u_{xxt} + \xi^2 u_{xtt}, \\
\varphi^{tt} &= D_t^2 Q + \xi^1 u_{xtt} + \xi^2 u_{ttt},
\end{aligned} \tag{6}$$

where the operators D_x and D_t denote the total derivatives with respect to x and t :

By theorem 6.5. in [?], the vector field X is a one parameter of Grad-Shafranov equation if and only if

$$\stackrel{(2)}{\text{Pr}} X(u_{xx} - \frac{1}{x}u_x + u_{tt} - x^2) = 0 \quad \text{whenever} \quad u_{xx} - \frac{1}{x}u_x + u_{tt} - x^2 = 0. \tag{7}$$

So, we apply the criterion of infinitesimal invariance in order to determine symmetries of the Grad-Shafranov equation[?]. Therefore, the infinitesimal symmetry criterion is

$$\varphi^{xx} - \frac{1}{x^2}\xi^1 u_x - \frac{1}{x}\varphi^x + \varphi^{tt} - 2x\xi^1 = 0 \tag{8}$$

Substituting the formulas (7) into (8), we are left with a polynomial equation involving the various derivatives of u whose coefficients are certain derivatives of ξ^1 , ξ^2 and φ only depend on x, t, u we can equate the individual coefficients to zero, leading to the complete set of defining equations:

$$\xi_u^2 = 0, \xi_{tt}^2 = -\frac{\xi_x^2}{x}, \quad \xi_{tx}^2 = 0, \quad \xi_{xx}^2 = \frac{\xi_x^2}{x}, \tag{9}$$

$$\varphi_{tt} = \frac{4\xi_t^2 x^3 + \varphi_x - \varphi_u x^3 - \varphi_{xx} x}{x}, \tag{10}$$

$$\varphi_{tu} = -\frac{\xi_x^2}{2x}, \quad \varphi_{uu} = 0, \quad \varphi_{ux} = 0, \quad \xi^1(x, t, u) = \xi_t^2 x. \tag{11}$$

By solving this system of PDEs, we state the following theorem.

Theorem *The Lie algebra \mathfrak{g} of the symmetry group G associated to the Grad-Shafranov equation is generated by the vector fields*

$$\begin{aligned}
X_1 &= \partial_t, \\
X_2 &= x\partial_x + t\partial_t + \frac{x^4}{2}\partial_u, \\
X_3 &= (-\frac{x^4}{8} + u)\partial_u, \\
X_4 &= tx\partial_x + \frac{1}{2}(t^2 - x^2)\partial_t + \frac{1}{16}t(7x^4 + 8u)\partial_u, \\
X_5 &= \psi(x, t)\partial_u.
\end{aligned} \tag{12}$$

The commutation relations of the 4-dimensional Lie algebra \mathfrak{g} spanned by the vector fields X_1, X_2, X_3, X_4 are shown in the following table.

Table 1: Commutation relations satisfied by infinitesimal generators in (33).

$[\cdot, \cdot]$	X_1	X_2	X_3	X_4
X_1	0	X_1	0	$X_2 + \frac{1}{3}X_3$
X_2	$-X_1$	0	0	X_4
X_3	0	0	0	0
X_4	$-X_2 - \frac{1}{3}X_3$	$-X_4$	0	0

3 The Lie algebra of symmetries

In this section, we determine the structure of full symmetry Lie algebra \mathfrak{g} of Eq. (3).

Theorem *The full symmetry Lie algebra \mathfrak{g} of Eq.(3) has the following semidirect decomposition:*

$$\mathfrak{g} = \mathbf{R} \ltimes \mathfrak{g}_1 \quad (13)$$

where \mathfrak{g}_1 is a semi-simple Lie algebra.

Proof: The center \mathbf{z} of \mathfrak{g} is $Span_{\mathbf{R}}\{X_3\}$. Therefore the quotient algebra \mathfrak{g}/\mathbf{z} is $Span_{\mathbf{R}}\{Y_1, Y_2, Y_3\}$, where $Y_i = X_i + \mathbf{z}$ for $i = 1, 2, 3$. The commutator table of this quotient algebra is given in the following table:

Table 2: Commutation relations satisfied by infinitesimal generators in (33).

$[\cdot, \cdot]$	Y_1	Y_2	Y_3
Y_1	0	Y_1	Y_2
Y_2	$-Y_1$	0	Y_3
Y_3	$-Y_2$	$-Y_3$	0

The Lie algebra \mathfrak{g} is non-solvable, because

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = Span_{\mathbf{R}}\{X_1, X_2 + \frac{1}{3}X_3, X_4\}, \quad (14)$$

$$\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \mathfrak{g}^{(1)}. \quad (15)$$

Similarly, \mathfrak{g}_1 is semi-simple and non-solvable, because

$$\mathfrak{g}_1^{(1)} = [\mathfrak{g}_1, \mathfrak{g}_1] = Span_{\mathbf{R}}\{Y_1, Y_2, Y_3\} = \mathfrak{g}_1. \quad (16)$$

The Lie algebra \mathfrak{g} admits a Levi decomposition as the following semi-direct product $\mathfrak{g} = r \ltimes s$, where $r = Span_{\mathbf{R}}\{X_3\}$ is the radical of \mathfrak{g} (the largest solvable ideal contained in \mathfrak{g}), and $s = Span_{\mathbf{R}}\{X_1, X_2 + \frac{1}{3}X_3, X_4\}$.

The ideal r is a one-dimensional subalgebra of \mathfrak{g} , therefore it is isomorphic to \mathbf{R} ; Thus the identity $\mathfrak{g} = r \ltimes s$ reduces to $\mathfrak{g} = \mathbf{R} \ltimes \mathfrak{g}_1$.

4 Optimal system of subalgebras

Consider a system of partial differential equations Δ defined over an open subset $M \subset X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$ of the space of independent and dependent variables. Let G be a local group of transformations acting on M . Roughly, a solution $u = f(x)$ of the system is said to be G -invariant if it is left unchanged by all the group transformations in G . If G is a symmetry group of a system of partial differential equations Δ , then, under some additional regularity assumptions on the action of G , we can find all the G -invariant solutions to Δ by solving a reduced system of differential equations, denoted by Δ/G , which will involve fewer independent variables than the original system Δ . In general, to each s -parameter subgroup H of the full symmetry group G of a system of differential equations in $p > s$ independent variables, there will correspond a family of group-invariant solutions to the system. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions to the system. We need an effective, systematic means of classifying these solutions, leading to an "optimal system" of group-invariant solutions from which every other such solution can be derived. Since elements $g \in G$ not in the subgroup H will transform an H -invariant solution to some other group-invariant solution, only those solution, not so related need be listed in our optimal system.

Let G be a Lie group. An optimal system of s -parameter subgroups is a list of conjugacy inequivalent s -parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s -parameter subalgebras forms an optimal system if every s -parameter subalgebra of \mathfrak{g} is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{\mathfrak{h}} = \text{Ad}g(\mathfrak{h}), g \in G$.

Proposition 3.7 of [?] says that the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by a nonzero vector in \mathfrak{g} . This problem is attacked by the naïve approach of taking a general element \mathbf{V} in \mathfrak{g} and subjecting it to various adjoint transformations so as to "simplify" it as much as possible. Thus we will deal with the construction of the optimal system of subalgebras of \mathfrak{g} .

The adjoint action is given by the Lie series

$$\text{Ad}(\exp(\varepsilon Y_i)Y_j) = Y_j - \varepsilon[Y_i, Y_j] + \varepsilon^2[Y_i, [Y_i, Y_j]] - \cdots, \quad (17)$$

where $[Y_i, Y_j]$ is the commutator for the Lie algebra, ε is a parameter and $i, j = 1, 2, 3$.

The adjoint representation of \mathfrak{g} is listed in the following table, it consists the separate adjoint actions of each element of \mathfrak{g} on all other elements. where the (i,j)-th entry indicating $\text{Ad}(\exp(\varepsilon Y_i)Y_j)$.

Table 3: Adjoint relations satisfied by infinitesimal generators in (33).

$[\cdot, \cdot]$	X_1	X_2	X_3	X_4
X_1	X_1	$X_2 + \varepsilon X_1$	X_2	$\frac{1}{2}\varepsilon^2 X_1 + \varepsilon X_2 + \frac{1}{2}\varepsilon X_3 + X_4$
X_2	$\exp(-\varepsilon)X_2$	X_2	X_3	$\exp(\varepsilon)X_4$
X_3	X_1	X_2	X_3	X_4
X_4	$X_1 - \varepsilon X_2 \frac{1}{2}\varepsilon X_3 + \frac{1}{2}\varepsilon^2 X_4$	$X_2 - \varepsilon X_1$	X_3	X_4

Theorem *An optimal system of one dimensional Lie subalgebras of Grad-Shafranov equation is provided by those generated by*

$$1) \ X_1, \ 2) X_2, \ 3) X_3, \ 4) X_3 - X_1, \ 5) X_3 + X_1 \quad (18)$$

$$6) \ aX_2 + X_3, \ 7) aX_1 + bX_2 + X_4. \quad (19)$$

where $a, b \in \mathbf{R}$ are arbitrary constants.

Proof Let \mathbf{g} is the symmetry group of Eq. (3) with adjoint representation determined in Table 2 and

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 \quad (20)$$

is a nonzero vector field of g . We will simplify as many of the coefficients of $a_i, i = 1, \dots, 4$; as possible through judicious applications of adjoint maps to X .

Case 1:

Suppose first that $a_4 \neq 0$. Scaling X if necessary, we can assume that $a_4 = 1$. Referring to table 2, if we act on such a X by $\text{Ad}(\exp(-a_3 X_1))$, we can make the coefficient of X_3 vanish. Thus, every one-dimensional subalgebra generated by a X with $a_4 \neq 0$ is equivalent to the subalgebra spanned by $aX_1 + bX_2 + X_4$ where $a, b \in \mathbf{R}$ are arbitrary constants. No further simplifications are possible.

Case 2:

The remaining one-dimensional subalgebras are spanned by vectors of the above form with $a_4 = 0$. If $a_3 \neq 0$, we can scale to make $a_3 = 1$. There are two subcases.

Case 2.1:

If $a_2 \neq 0$, then we can cancel the coefficient of X_1 by acting on X by $\text{Ad}(\exp(-\frac{a_1}{a_2} X_1))$. So that, X is equivalent to a scalar multiple $aX_2 + X_3$ for some $a \in \mathbf{R}$.

Case 2.2:

If $a_2 = 0$, we can act by adjoint map generated by X_2 to arrange the coefficient of X_1 either +1, -1 or 0. Therefore, any one-dimensional subalgebra spanned by X with $a_4 = 0, a_3 = 1, a_2 = 0$ is equivalent to one spanned by either $X_3 + X_1, X_3 - X_1$ or X_3 .

Case 3:

The remaining cases, $a_3 = a_4 = 0$, are similarly seen to be equivalent either to $X_2(a_2 \neq 0)$ or to $X_1(a_2 = a_3 = a_4 = 0)$.

There is not any more possible case for studying and the proof is complete.

In continuation, we find some group invariant solutions of the equation(??) corresponding to 1-dimensional subalgebras generated by X_1, X_2, X_4 .

Consider

$$X_2 = x\partial_x + t\partial_t + \frac{x^4}{2}\partial_u \quad (21)$$

invariants are

$$C_1 = \frac{t}{x}, \quad C_2 = -\frac{1}{8}x^4 + u \quad (22)$$

So, group invariant solution associated to these invariants has the form

$$u(x, t) = \frac{(\frac{x^4}{8} + C_1)\sqrt{t^2 + x^2} + C_2 t}{\sqrt{t^2 + x^2}} \quad (23)$$

In the case of X_4 , invariants are

$$C_1 = \frac{t^2 + x^2}{x}, \quad C_2 = \frac{8u - x^4}{8\sqrt{x}} \quad (24)$$

therefore, we get the associated group invariant solution

$$\frac{\frac{1}{8}(x^5 t^2 + x^7)\sqrt{\frac{t^2 + x^2}{x}} + (2C_2 t^2 + C_1)x^{5/2} + C_2(x^{9/2} + \sqrt{x}t^4)}{\sqrt{\frac{t^2 + x^2}{x}}x(t^2 + x^2)} \quad (25)$$

In the case of $X = X_2 + X_3$, we obtain the following invariants

$$C_1 = \frac{t}{x}, \quad C_2 = \frac{8u - x^4}{8x} \quad (26)$$

Thus corresponding invariant solution is

$$u(x, t) = \frac{x^4}{8} + C_2 \sqrt{\frac{t^2 + x^2}{x^2}}x + C_1 t \quad (27)$$

In the case of $X = X_1 + X_3$, we get the invariants

$$C_1 = x, \quad C_2 = \frac{-1}{8}(x^4 - 8u)\exp(-t) \quad (28)$$

therefore, we obtain the following invariant solution associated to these invariants.

$$u(x, t) = x\left(\frac{1}{8}x^3 + C_1 \exp(t)\text{BesselJ}(1, x) + C_2 \exp(t)\text{BesselY}(1, x)\right) \quad (29)$$

where BesselJ and BesselY are the Bessel functions of the first and second kinds, respectively. They satisfy Bessel's equation.

For the vector field $X_1 = \partial_t$, global invariants are $C_1 = x$, $C_2 = u$. So, the solution of reduced equation is

$$u = x^4 + 4C_1x^2 + C_2 \quad (30)$$

5 New admitted symmetries

In this section, we probe the remainder cases.

If $F = \exp(2u)$ and $G = \exp(u)$, there is a new admitted symmetry which has the form

$$X = x\partial_x + t\partial_t - 2\partial_u \quad (31)$$

If $F = u^{1+\frac{2}{q}}$ and $G = u^{1+\frac{1}{q}}$ for all $q \neq 0$, there is a new admitted symmetry which is

$$X = x\partial_x + t\partial_t - 2qu\partial_u \quad (32)$$

If $F = 1, G = u$, new symmetries are

$$X = (x^2 + u)\partial_u, \quad X = \psi(x, t)\partial_u \quad (33)$$

Furthermore, the associated invariant solution for the global invariants $C_1 = x$, $C_2 = u$ is

$$u(x, t) = -x(x - C_2 \text{BesselI}(1, x) + C_1 \text{BesselK}(1, x)) \quad (34)$$

where BesselI and BesselK are the modified Bessel functions of the first and second kinds, respectively which satisfy the modified Bessel equation.

If $F = u, G = 1$, the new symmetry has the form

$$X = (u + \psi(x, t))\partial_u \quad (35)$$

In addition, the corresponding invariant solution for the global invariants $C_1 = x$, $C_2 = u$ is

$$u(x, t) = \cosh\left(\frac{x^2}{2}\right)(C_1 - \frac{1}{2}\text{Shi}\left(\frac{x^2}{2}\right)) + \sinh\left(\frac{x^2}{2}\right)(C_2 + \frac{1}{2}\text{Chi}\left(\frac{x^2}{2}\right)) \quad (36)$$

where Shi and Chi are hyperbolic Sine integral and hyperbolic cosine integral, respectively.

$$\text{Shi}(x) = \int_0^x \frac{\sinh(t)}{t} dt \quad (37)$$

$$\text{Chi}(x) = \gamma + \ln(x) + \int_0^x \frac{\cosh(t) - 1}{t} dt \quad (38)$$

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